

Merge Sort

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DAINF-UTFPR-PG

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Este material é preparado usando como referências os textos dos seguintes livros.

Divisão e Conquista é uma técnica de projeto de algoritmos que consiste em dividir o problema original em subproblemas, resolver esses subproblemas e combinar essas soluções para obter a solução do problema original.

Dica: Normalmente, os resultados são melhores se você divide o problema original em subproblemas de tamanhos parecidos (iguais, se possível).

Exemplo de algoritmo de Divisão e Conquista: MergeSort ou ordenação por intercalação.

Divisão e Conquista: MergeSort

Você se lembra de como funciona o MergeSort?

Divisão e Conquista: MergeSort

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Divisão e Conquista: MergeSort

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Divisão e Conquista: MergeSort

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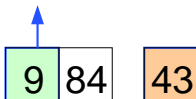
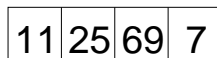
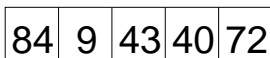
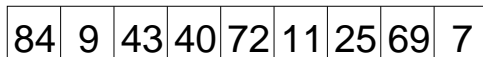
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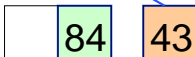
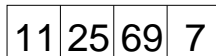
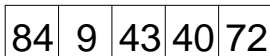
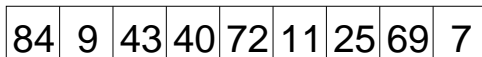
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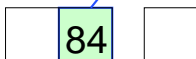
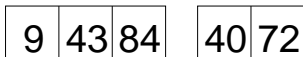
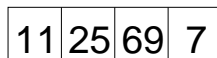
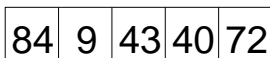
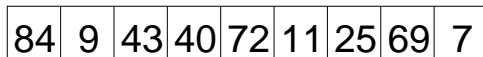
Divisão e Conquista: MergeSort



Divisão e Conquista: MergeSort



Divisão e Conquista: MergeSort



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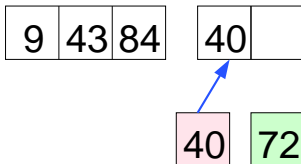
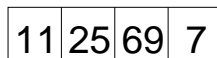
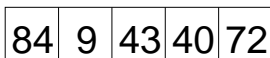
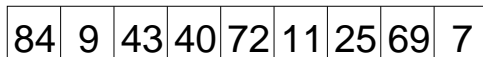
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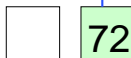
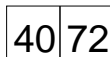
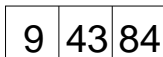
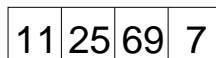
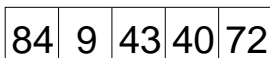
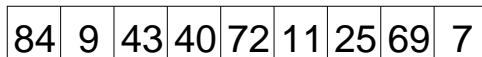
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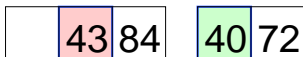
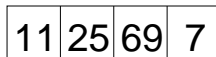
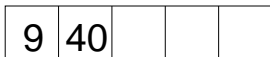
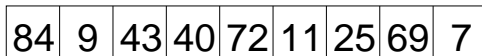
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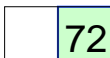
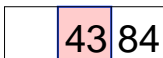
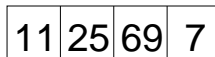
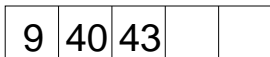
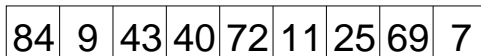
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Divisão e Conquista: MergeSort



Divisão e Conquista: MergeSort



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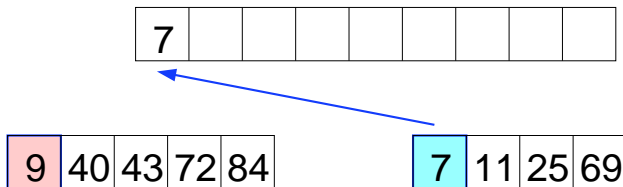
Divisão e Conquista: MergeSort

| | | | | | | | | |
|----|---|----|----|----|----|----|----|---|
| 84 | 9 | 43 | 40 | 72 | 11 | 25 | 69 | 7 |
|----|---|----|----|----|----|----|----|---|

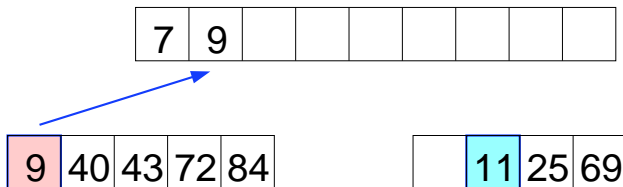
| | | | | |
|---|----|----|----|----|
| 9 | 40 | 43 | 72 | 84 |
|---|----|----|----|----|

| | | | |
|---|----|----|----|
| 7 | 11 | 25 | 69 |
|---|----|----|----|

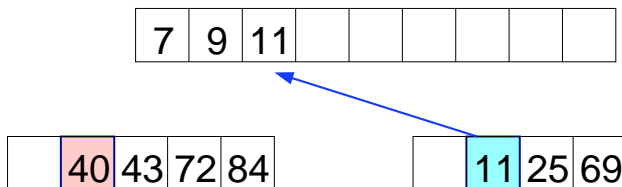
Divisão e Conquista: MergeSort



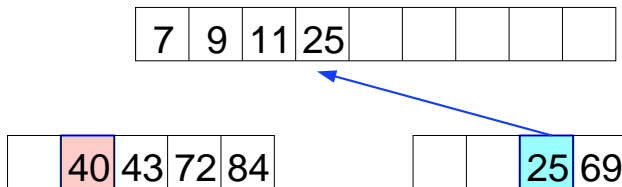
Divisão e Conquista: MergeSort



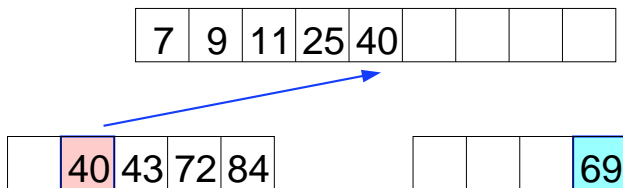
Divisão e Conquista: MergeSort



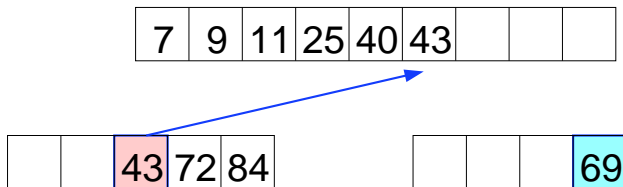
Divisão e Conquista: MergeSort



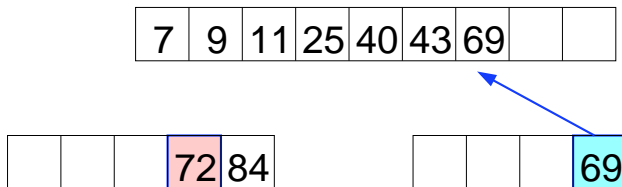
Divisão e Conquista: MergeSort



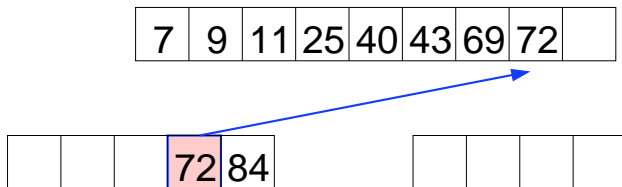
Divisão e Conquista: MergeSort



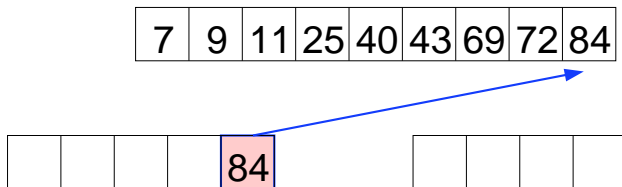
Divisão e Conquista: MergeSort



Divisão e Conquista: MergeSort



Divisão e Conquista: MergeSort



Divisão e Conquista: MergeSort

| | | | | | | | | |
|---|---|----|----|----|----|----|----|----|
| 7 | 9 | 11 | 25 | 40 | 43 | 69 | 72 | 84 |
|---|---|----|----|----|----|----|----|----|

Qual a complexidade do MergeSort?

Merge_Sort($V[], p, r$)

Se $p < r$, então:

$$q \leftarrow \lfloor \frac{r+p}{2} \rfloor$$

Merge_Sort(V, p, q)

Merge_Sort($V, q + 1, r$)

Merge(V, p, q, r)

Divisão e Conquista: MergeSort

Merge_Sort($V[], p, r$)

Se $p < r$, então: **1**

$q \leftarrow \lfloor \frac{r+p}{2} \rfloor$ **3**

Merge_Sort(V, p, q) $T(\frac{n}{2})$

Merge_Sort($V, q + 1, r$) $T(\frac{n}{2})$

Merge(V, p, q, r) $M(n)$

Divisão e Conquista: MergeSort

Merge($V[], p, q, r$)

$i \leftarrow p; j \leftarrow q + 1; h \leftarrow 0; 4$

Aloque vetor temporário $W[0..r - 1]$; assumindo que uma página de memória é suficiente, muito barato

Enquanto $i \leq q$ e $j \leq r$ faça: $2n$ (verdadeiro $n-1$ vezes)

Se $V[i] \leq V[j]$ então: $n-1$

$W[h] \leftarrow V[i]; i \leftarrow i + 1; 3(n - 1)$

Senão:

$W[h] \leftarrow V[j]; j \leftarrow j + 1; rodou a de cima, não roda nesta$

$h \leftarrow h + 1; 2(n - 1)$

Enquanto $i \leq q$ faça: 2

$W[h] \leftarrow V[i]; h \leftarrow h + 1; i \leftarrow i + 1; 5$

Enquanto $j \leq r$ faça: 1 (falso aqui ou no de cima)

$W[h] \leftarrow V[j]; h \leftarrow h + 1; j \leftarrow j + 1; 0$

Para i de p até r faça: $3(n + 1) - 1 = 3n + 2$

$V[i] \leftarrow W[i - p]; 2n$

Complexidade da função Merge() em um vetor com n elementos:

$$M(n) = 13n + 7$$

Divisão e Conquista: MergeSort

Merge_Sort($V[], p, r$)

Se $p < r$, então: **1**

$q \leftarrow \lfloor \frac{r+p}{2} \rfloor$ **3**

Merge_Sort(V, p, q) $T(\frac{n}{2})$

Merge_Sort($V, q + 1, r$) $T(\frac{n}{2})$

Merge(V, p, q, r) $13n + 7$

Divisão e Conquista: MergeSort

Complexidade do Algoritmo Merge_Sort: análise de tempo em um vetor com n elementos, onde: $r - p + 1 = n$.

$$T(n) = \begin{cases} 1, & \text{se } n = 1; \\ 2T(\lceil \frac{n}{2} \rceil) + 13n + 7, & \text{se } n > 1. \end{cases}$$

Divisão e Conquista: MergeSort

Solução da relação de recorrência.

| n | $T(n)$ |
|------------|---|
| $1 = 2^0$ | 1 |
| $2 = 2^1$ | $2T(1) + 13(2) + 7 = 2 + 13(2) + 7$ |
| $4 = 2^2$ | $2T(2) + 13(2^2) + 7$ $= 2^2 + (13)2^2 + 2(7) + 13(2^2) + 7$ $= 2^2 + 2(2^2)(13) + 2(7) + 7 = 2^2 + 2(2^2)(13) + 2(7) + 7$ |
| $8 = 2^3$ | $2T(4) + 13(2^3) + 7$ $= 2^3 + 2(2^3)(13) + 2^2(7) + 2(7) + 13(2^3) + 7$ $= 2^3 + 3(2^3)(13) + 2^2(7) + 2(7) + 7$ |
| $16 = 2^4$ | $2T(8) + 13(2^4) + 7$ $= 2^4 + 3(2^4)(13) + 2^3(7) + 2^2(7) + 2(7) + 13(2^4) + 7$ $= 2^4 + 4(2^4)(13) + 2^3(7) + 2^2(7) + 2(7) + 7$ |
| 2^k | $2^k + k(2^k)(13) + 2^{k-1}(7) + 2^{k-2}(7) + \dots + 2(7) + 7$ $= 2^k + k(2^k)(13) + \sum_{i=0}^{k-1} (2^i(7))$ |

Divisão e Conquista: MergeSort

Solução da relação de recorrência.

$$T(2^k) = 2^k + k(2^k)(13) + \sum_{i=0}^{k-1} (2^i(7))$$

$$T(2^k) = 2^k + k(2^k)(13) + 7 \sum_{i=0}^{k-1} (2^i) = 2^k + k(2^k)(13) + 7(2^k - 1)$$

$$T(2^k) = (1 + 13k)2^k + 7(2^k) - 7 = (13k + 8)2^k - 7$$

Quando $2^k = n$ temos $k = \log_2 n$,

$$T(n) = (13 \log_2 n + 8)n - 7 = 13n \log_2 n + 8n - 7$$